Chern classes = characteristic classes of complex vector bundles

ie. $E \xrightarrow{\pi} B$ where fibers are $\mathbb{C}$-vector spaces, change of triv$^{ns}$ are given by fns. $U_i \cap U_j \xrightarrow[g_{ij}]{} GL(n, \mathbb{C})$

- Note: a $\mathbb{C}$-vector space carries a natural orientation
(take a $\mathbb{R}$-basis $e_1, ie_1, e_2, ie_2, \dots$ to be positively oriented, where $e_1 \dots e_n$ $\mathbb{C}$-basis)
Hence: viewing $E$ as an oriented rank $2n$ real vector bundle,
we have the Euler class $e(E) \in H^{2n}(B, \mathbb{Z})$.
This will be the starting point of inductive def. of Chern classes.
- Natural metrics: Hermitian metrics on a $\mathbb{C}$-vector bundle:
= Eucl. metric st. $|iv| = |v|$. $\quad (|v|^2 = v \cdot v)$
(this implies $(iv)\cdot(iw) = v \cdot w$, and hence $v \cdot (iv) = 0$).
ie. $\langle v, w \rangle := v \cdot w + i\, v \cdot (iw)$ defines a Hermitian scalar product,
$\mathbb{C}$-linear in $v$, $\mathbb{C}$-antilinear in $w$,
$\langle v, w \rangle = \overline{\langle w, v \rangle}$, $\langle v, v \rangle = |v|^2$.
$\rightsquigarrow$ orthogonality, ....
If $B$ paracompact then every $\mathbb{C}$ vect. bundle admits a Herm. metric.

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Construction: $E \xrightarrow[\pi]{} B$ rank $n$ $\mathbb{C}$-vect bundle. Equip it w/ a Herm. metric

$E_0 = E - B$ nonzero vectors, $\exists$ canonical rank $(n-1)$ $\mathbb{C}$-vect bundle $E' \to E_0$

namely: $\pi_0^* E \to E_0$ has a canonical nowhere vanishing section:
at $p = (x, v) \in E_0$ $\quad (v \in E_x, \ v \neq 0)$, take $v$ itself $\in (\pi_0^* E)_p = E_x$.
This spans a tautological line subbundle $\tau$; we set
$E' = \tau^\perp \subset \pi_0^* E$ ie. fiber at $p = (x, v)$ is $v^\perp \subset E_x$.
(Or, without choosing a metric, $E' = \pi_0^* E / \tau$, with fiber $E_x / \mathbb{C}v$.)

(Rmk: instead of $E_0$, could use $V_1(E)$ = unit vectors in $E$)

Now recall: $\dots \to H^i(E, E_0) \to H^i(E) \to H^i(E_0) \to H^{i+1}(E, E_0) \to \dots$

$$H^{i-2n}(B) \xrightarrow[\text{Thom}]{\pi^* u \cup} H^i(E, E_0), \qquad H^i(B) \underset{\pi^*}{\cong} H^i(E), \qquad H^{i+1-2n}(B) \underset{\text{Thom}}{\cong} H^{i+1}(E, E_0)$$

so for $i < 2n-1$, the projection $E_0 \xrightarrow[\pi_0]{} B$ induces an isom. $H^i(B, \mathbb{Z}) \underset{\pi_0^*}{\xrightarrow{\sim}} H^i(E_0, \mathbb{Z})$

**Def.:** By induction on the rank $n$ of $E \to B$, define by induction:

$$c_n(E) := e(E) \in H^{2n}(B, \mathbb{Z})$$

$$c_i(E) := (\pi_0^*)^{-1}\big(c_i(E')\big) \in H^{2i}(B, \mathbb{Z}).$$
$$\underset{H^{2i}(E_0, \mathbb{Z})}{\cap}$$

(by def., $c_0(E) = 1 \in H^0$, $c_i(E) = 0$ for $i > n$.)

Total Chern class: $c(E) = 1 + c_1(E) + \dots + c_n(E) \in H^*(B, \mathbb{Z})$

---

Properties:

- Naturality: $\begin{array}{ccc} \tilde{E} = f^*E & \xrightarrow{\tilde{f}} & E \\ \downarrow & & \downarrow \\ f: A & \longrightarrow & B \end{array}$ $\Rightarrow c_i(\tilde{E}) = f^* c_i(E)$

  Pf.: Induction on rank$(E)$:
  - naturality of $c_n$ follows from that of Euler class.
  - for $i < n$, assuming result known for rank $n-1$ bundles: note the constr. above is natural, namely

$$\tilde{\pi}_0^* \tilde{E} = \tilde{f}^* \pi_0^* E \cong \tilde{f}^* E' \oplus \tau \qquad \pi_0^* E \cong E' \oplus \tau$$

$$\begin{array}{ccc} \tilde{E}_0 & \xrightarrow{\tilde{f}} & E_0 \\ \tilde{\pi}_0 \downarrow & & \downarrow \pi_0 \\ A & \xrightarrow{f} & B \end{array}$$

so $(\tilde{E})' \simeq \tilde{f}^*(E')$ as $\mathbb{C}^{n-1}$-bundles on $\tilde{E}_0$

so $c_i((\tilde{E})') = \tilde{f}^* c_i(E')$; via isoms. $\tilde{\pi}_0^*$, $\pi_0^*$ and using commutativity of above diagram, $c_i(\tilde{E}) = f^* c_i(E)$ ▲

- Lemma: $\underline{\mathbb{C}}^r$ trivial rank $r$ cx. vector bundle : $\mathbb{C}^r \times B \to B$, then
$$c_i(E \oplus \underline{\mathbb{C}}^r) = c_i(E) \quad \forall i.$$

  Pf.: by induction, case $r = 1$ sufficient. If $rk_{\mathbb{C}}(E) = n$: let $\tilde{E} = E \oplus \underline{\mathbb{C}}$
  - $c_{n+1}(\tilde{E}) = e(E \oplus \underline{\mathbb{C}}) = 0$ since $\exists$ nonvanishing section $s = (0, 1)$.
  - $s: B \to \tilde{E}_0$ induces isom. $s^*(\tilde{E}') \simeq E$ since $\tau = \mathrm{span}(s)$

    so $\tilde{E}'_{s(x)} = \mathrm{span}(s)^\perp \simeq E_x$

    so $c_i(E) = s^* c_i(\tilde{E}') = s^* \pi_0^* c_i(\tilde{E})$
    $= (\pi_0 \circ s)^* c_i(\tilde{E}) = c_i(\tilde{E}).$ ▲

## Complex Grassmannians & tautological bundles

$G_n(\mathbb{C}^{n+k})$ = Grassmannian of complex $n$-planes in $\mathbb{C}^{n+k}$

(special case: cx. projective space $G_1(\mathbb{C}^{k+1}) = \mathbb{CP}^k$ = lines in $\mathbb{C}^{k+1}$).

carries a tautological rank $n$ complex vector bundle $\tau \to G_n(\mathbb{C}^{n+k})$

(subbundle of trivial rank $n+k$ bundle whose fiber at $v \in G_n$ is the $n$-dim'l subspace $v$ itself).

## Cohomology of $G_n(\mathbb{C}^\infty)$:

**Thm:** $H^*(G_n(\mathbb{C}^\infty), \mathbb{Z}) \overset{\text{ring}}{\cong} \mathbb{Z}[c_1, \dots, c_n]$, $c_i = c_i(\tau) \in H^{2i}(G_n(\mathbb{C}^\infty), \mathbb{Z})$ (free polynomial algebra)

- first the case $n=1$: $\tau \to \mathbb{CP}^k = G_1(\mathbb{C}^{k+1})$ tautological line bundle.

$$\tau_0 \cong \mathbb{C}^{k+1} \setminus \{0\} \longrightarrow \mathbb{CP}^k$$
$$v \longmapsto [v] = \text{line } \mathbb{C}\cdot v \qquad \text{so } \tau_0 \text{ is } 2k\text{-connected}$$

so:
$$\begin{array}{ccccccccc}
\dots H^{i-1}(\tau_0) & \to & H^i(\tau, \tau_0) & \to & H^i(\tau) & \to & H^i(\tau_0) & \to & \\
 & & \text{Thom}\uparrow\simeq & & \pi^*\uparrow\simeq & & \uparrow & & \\
\dots H^{i-1}(\tau_0) & \to & H^{i-2}(\mathbb{CP}^k) & \xrightarrow[\cup c_1(\tau)]{} & H^i(\mathbb{CP}^k) & \to & H^i(\tau_0) & \to & \dots
\end{array}$$
Gysin seq.

$\Rightarrow$ for $i \leq 2k$, $H^{i-2}(\mathbb{CP}^k) \xrightarrow[\cup c_1(\tau)]{\cong} H^i(\mathbb{CP}^k)$ is an isom.

Hence $H^*(\mathbb{CP}^k, \mathbb{Z}) \underset{\text{ring}}{\cong} \mathbb{Z}[c]/c^{k+1}$, $c = c_1(\tau) \in H^2(\mathbb{CP}^k, \mathbb{Z})$

and taking $k \to \infty$, $H^*(\mathbb{CP}^\infty, \mathbb{Z}) \simeq \mathbb{Z}[c]$. ✓

- general case: <u>by induction on $n$</u>: assume result known for $G_{n-1}(\mathbb{C}^\infty)$.

Consider $\tau_n \longrightarrow G_n(\mathbb{C}^{n+k})$ ; $\exists$ natural map $\tau_n^0 \xrightarrow{f} G_{n-1}(\mathbb{C}^{n+k})$

$$(v, u) \longmapsto v' = u^\perp \cap v \subset \mathbb{C}^{n+k}$$
$v \subset \mathbb{C}^{n+k}$, $\dim v = n$ $\qquad \dim v' = n-1$.
$u \in v$, $u \neq 0$

This is a fiber bundle with fiber $f^{-1}(v') = \left\{ (v' \oplus \mathbb{C}u, u) \,/\, \begin{array}{l} u \in v'^\perp \\ u \neq 0 \end{array} \right\}$
$\simeq \mathbb{C}^{k+1} - \{0\}$

In fact $\tau_n^0 \simeq (\tau_{n-1}^\perp)^0$ via $\{(v,u) / u \neq 0, u \in v\} \underset{v = v' \oplus \mathbb{C}u}{\longleftrightarrow} \{(v', u) / u \neq 0, u \in v'^\perp\}$

$$\begin{tikzcd}
 & \tau_n^0 \arrow[dl, "\pi"'] \arrow[r, phantom, "\simeq"] \arrow[drr, bend left, "f"] & (\tau_{n-1}^\perp)^0 \arrow[dr] & \\
G_n(\mathbb{C}^{n+k}) & & & G_{n-1}(\mathbb{C}^{n+k})
\end{tikzcd}$$

Moreover, $\pi^*(\tau_n) \simeq f^*(\tau_{n-1}) \oplus \underline{\mathbb{C}}$ as rank $n$ cx-bundles over $\tau_n^0$

fiber at $(v,u) = v$     fiber at $(v,u) = v' \oplus \mathbb{C}u = v$

* For $i \leq 2k$, Gysin seq. of $\mathbb{C}^{k+1} \setminus \{0\}$-bundle $\tau_n^0 \simeq (\tau_{n-1}^\perp)^0 \xrightarrow[f]{} G_{n-1}(\mathbb{C}^{n+k})$ gives

$$\begin{array}{ccccccc} \cdots H^i(\tau_{n-1}^\perp, \tau_{n-1}^{\perp 0}) & \to & H^i(\tau_{n-1}^\perp) & \to & H^i((\tau_{n-1}^\perp)^0) & \to & H^{i+1}(\tau_{n-1}^\perp, \tau_{n-1}^{\perp 0}) \cdots \\ \wr\!\wr \text{ Thom} & & \wr\!\wr & \nearrow f^* & & & \wr\!\wr \text{ Thom} \\ 0 & & H^i(G_{n-1}(\mathbb{C}^{n+k})) & & & & 0 \end{array}$$

hence $f^*: H^i(G_{n-1}(\mathbb{C}^{n+k})) \to H^i(\tau_n^0)$ is an isomorphism

and, since $\pi^*(\tau_n) \simeq f^*(\tau_{n-1}) \oplus \underline{\mathbb{C}}$, by naturality, $\pi^* c_j(\tau_n) = f^* c_j(\tau_{n-1})$.

* Inserting the isom. $f^*$ into Gysin seq. of $\tau_n^0 \xrightarrow[\pi]{} G_n(\mathbb{C}^{n+k})$ we get:

for $i \leq 2k$,

$$\begin{array}{ccccccc} \cdots \to H^i(\tau_n, \tau_n^0) & \to & H^i(\tau_n) & \to & H^i(\tau_n^0) & \to & H^{i+1}(\tau_n, \tau_n^0) \to \cdots \\ \wr\!\wr \text{ Thom} & & \wr\!\wr \pi^* & & \wr\!\wr f^* & & \\ \cdots \to H^{i-2n}(G_n(\mathbb{C}^{n+k})) & \xrightarrow[\cup c_n(\tau_n)]{} & H^i(G_n(\mathbb{C}^{n+k})) & \xrightarrow{\lambda} & H^i(G_{n-1}(\mathbb{C}^{n+k})) & \to & \cdots \end{array}$$

$\uparrow$ this maps $c_j(\tau_n)$ to $c_j(\tau_{n-1})$

* Taking $k \to \infty$, get similar sequence for $\infty$ Grassmannians, without restr. on $i$.

$$\cdots \to H^{i-2n}(G_n(\mathbb{C}^\infty)) \xrightarrow[\cup c_n]{} H^i(G_n(\mathbb{C}^\infty)) \xrightarrow[\lambda]{} H^i(G_{n-1}(\mathbb{C}^\infty)) \to \cdots$$

$$\lambda(\prod c_j^{k_j}) = \prod c_j^{k_j}$$

By induction hypothesis, $H^*(G_{n-1}(\mathbb{C}^\infty))$ is generated as ring by $c_j(\tau_{n-1})$, so $\lambda$ is surjective !! Hence l.e.s. splits into short exact sequences

$$0 \to H^{i-2n}(G_n(\mathbb{C}^\infty)) \underset{\cup c_n(\tau_n)}{\hookrightarrow} H^i(G_n(\mathbb{C}^\infty)) \overset{\lambda}{\twoheadrightarrow} H^i(G_{n-1}(\mathbb{C}^\infty)) \to 0$$

$\forall x \in H^i(G_n(\mathbb{C}^\infty))$, write $\lambda(x) = p(c_1(\tau_{n-1}) \ldots c_{n-1}(\tau_{n-1}))$ for some polynomial $p$

then get unique decomp. $x = x' \cdot c_n(\tau_n) + p(c_1(\tau_n), \ldots, c_{n-1}(\tau_n))$. $\quad (*)$

$\leadsto$ proceeding by induction on $\deg(x)$, express $x$ uniquely as a polynomial in $c_1(\tau_n), \ldots, c_{n-1}(\tau_n), c_n(\tau_n)$.

(and $*$ is division algorithm by last indeterminate).

$\blacktriangle$

As in real case, given $E \to B$ rank $n$ cx. vect bundle, if we have a map $\hat{g}: E \longrightarrow \mathbb{C}^{n+k}$ st. restr. $\hat{g}_{|E_x}: E_x \to \mathbb{C}^{n+k}$ linear injective $\forall x \in B$

then this induces $g: B \to G_n(\mathbb{C}^{n+k})$
$x \longmapsto \mathrm{Im}(\hat{g}_{|E_x})$ with $E \simeq g^* \tau$.

Can do this for compact $B$ by picking local trivializations etc...; in general, need infinite limit

$G_n(\mathbb{C}^\infty) = \lim_{k\to\infty} G_n(\mathbb{C}^{n+k})$, $\tau \to G_n(\mathbb{C}^\infty)$ tautological bundle;

<u>Thm</u>: $E \to B$ rank $n$ $\mathbb{C}$-vect bundle over paracompact $B$ $\Rightarrow$ $\exists$ classifying map $g: B \to G_n(\mathbb{C}^\infty)$, unique up to homotopy, st. $E \simeq g^* \tau$.

---

<u>Product theorem for Chern classes</u>: $c(E \oplus F) = c(E) \cdot c(F)$

<u>Pf</u>: Can use classifying maps $B \xrightarrow[g_E]{} G_n(\mathbb{C}^\infty)$, $B \xrightarrow[g_F]{} G_m(\mathbb{C}^\infty)$

$$\Rightarrow \begin{array}{ccc} E \oplus F & \longrightarrow & \tau_n \times \tau_m = p_1^* \tau_n \oplus p_2^* \tau_m \\ \downarrow & & \downarrow \\ B & \xrightarrow[\bar{g} = (g_E, g_F)]{} & G_n(\mathbb{C}^\infty) \times G_m(\mathbb{C}^\infty) \end{array}$$

By naturality, enough to check result for universal bundles! Namely:

$$c(\tau_n \times \tau_m) = p_{n,m} \in H^*(G_n(\mathbb{C}^\infty) \times G_m(\mathbb{C}^\infty), \mathbb{Z}) \simeq \mathbb{Z}[c_1 \dots c_n] \otimes \mathbb{Z}[c_1' \dots c_m']$$
$$\simeq \mathbb{Z}[c_1, \dots, c_n, c_1' \dots c_m'].$$

Then we'll have; $\forall E, F$ of ranks $n$ and $m$, ("universal" polynomial formula)

$$c(E \oplus F) = \bar{g}^*(p_{n,m}(c_1 \dots c_n, c_1' \dots c_m')) = p_{n,m}(c_1(E), \dots, c_n(E), c_1(F), \dots, c_m(F))$$

Want to show: $p_{n,m} = (1 + c_1 + \dots + c_n)(1 + c_1' + \dots + c_m')$

We do this by induction on $m+n$: assume result true for lower rank.

- for $(p_1^* \tau_{n-1} \oplus \underline{\mathbb{C}}) \oplus p_2^* \tau_m \to G_{n-1}(\mathbb{C}^\infty) \times G_m(\mathbb{C}^\infty)$, in $H^*(G_{n-1} \times G_m) \simeq \mathbb{Z}[c_1 \dots c_{n-1}, c_1' \dots c_m']$,

$$c(p_1^* \tau_{n-1} \oplus \underline{\mathbb{C}} \oplus p_2^* \tau_m) = p_{n,m}(\underbrace{c_1, \dots, c_{n-1}, 0}_{\text{Chern class of } p_1^* \tau_{n-1} \oplus \underline{\mathbb{C}}}; \underbrace{c_1' \dots c_m'}_{\text{\& of } p_2^* \tau_m})$$

$$\| $$

$$c(p_1^* \tau_{n-1} \oplus p_2^* \tau_m) = p_{n-1,m}(c_1 \dots c_{n-1}; c_1' \dots c_m') \underset{\text{induction}}{=} (1 + c_1 + \dots + c_{n-1})(1 + c_1' + \dots + c_m').$$

Since $c_1 \dots c_{n-1}, c'_1 \dots c'_m$ alg. independent, this implies that as abstract polynomials in $m+n$ variables,

$$P_{n,m} \equiv (1+c_1+\dots+c_{n-1})(1+c'_1+\dots+c'_m) \mod c_n.$$
$$\equiv (1+c_1+\dots+c_n)(1+c'_1+\dots+c'_m) \mod c_n.$$

- similarly by considering $G_m \times G_{n-1}$,

$$P_{n,m} \equiv (1+c_1+\dots+c_n)(1+c'_1+\dots+c'_m) \mod c'_m.$$

Since $\mathbb{Z}[c_1,\dots,c_n,c'_1\dots c'_m]$ unique factorization domain, this implies

$$P_{n,m} = (1+c_1+\dots+c_n)(1+c'_1+\dots+c'_m) + u\, c_n c'_m \quad \text{for some } u.$$

Since $c(E\oplus F)$ has no term of cohomological degree $> 2(m+n)$, necess. $\deg(u)=0$
$\Rightarrow u =$ constant $\in \mathbb{Z}$.

But $c_{n+m}(E\oplus F) = e(E\oplus F) = e(E).e(F) = c_n(E).c_m(F) \Rightarrow u=0$. ∎

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**More properties:**

- Complex conjugate: $\overline{E} = E$ as underlying real vector bundle, but w/ conjugate complex structure, ie. mult. by $i$ in $E \longleftrightarrow$ mult by $-i$ in $\overline{E}$

in other terms, $\exists$ $\mathbb{C}$-antilinear isomorphism $f: E \xrightarrow{\sim} \overline{E}$ ($f=$"id")
$f(\lambda v) = \bar{\lambda} f(v)\ \forall \lambda\in\mathbb{C}\ \forall v\in E$.

In other terms: $(v_1, w_1=iv_1,\dots, v_n, w_n=iv_n)$ $\mathbb{R}$-basis of $E_x$, $(v_1\dots v_n)$ $\mathbb{C}$-basis
$\Rightarrow (v_1, w_1\dots v_n, w_n)$ also $\mathbb{R}$-basis of $\overline{E}_x = E_x$
but now w/ $\mathbb{C}$ structure $(a+ib)v_n = av_n - ibw_n$.

Cor: natural orientation of $E$ and $\overline{E}$ differ by $(-1)^n$!
indeed: $(v_1, w_1, \dots, v_n, w_n)$ oriented basis of $E_x$ for preferred orient.
$(v_1, -w_1, \dots, v_n, -w_n)$ —"— of $\overline{E}_x$

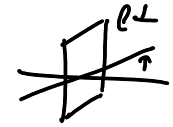
Lemma: $\| \ c_j(\overline{E}) = (-1)^j c_j(E).$

Pf: for $c_n$ this follows from behavior of Euler class under orientation change.
for $j<n$: by induction, observing that pullback of $\overline{E}$ to $\overline{E}_0 = E_0$ is the complex conjugate of the pullback of $E$ to $E_0$.

- Dual: $E^* = \mathrm{Hom}_{\mathbb{C}}(E, \underline{\mathbb{C}})$: same answer, since a Herm. metric on $E$ induces a $\mathbb{C}$-antilinear isom. from $E$ to $E^*$, $v \mapsto \langle\cdot, v\rangle \in \mathrm{Hom}_{\mathbb{C}}(E_x,\mathbb{C})$, so $E^* \simeq \overline{E}$.

<u>Example:</u> $c(T\mathbb{C}P^n) = (1+a)^{n+1}$ where $a = -c_1(\tau) \in H^2(\mathbb{C}P^n, \mathbb{Z})$ generator
$= 1 + (n+1)a + \dots + (n+1)a^n$.

Indeed: same argument as in real case $\Rightarrow$ at $x \in \mathbb{C}P^n$ representing a line $\ell \subset \mathbb{C}^{n+1}$,
$T_x\mathbb{C}P^n \underset{\text{canonical}}{\cong} \hom_{\mathbb{C}}(\ell, \ell^\perp)$ (since open nbd of $x$ = lines which are graphs of such maps,



)

So $T_x\mathbb{C}P^n \oplus \mathbb{C} \cong \hom(\ell, \ell^\perp) \oplus \hom(\ell,\ell) \cong \hom(\ell, \mathbb{C}^{n+1})$

$\rightsquigarrow T\mathbb{C}P^n \oplus \underline{\mathbb{C}} = \underbrace{\tau^* \oplus \dots \oplus \tau^*}_{n+1}$

$\Rightarrow c(T\mathbb{C}P^n) = c(T\mathbb{C}P^n \oplus \underline{\mathbb{C}}) = (1 + c_1(\tau^*))^{n+1} = (1 - c_1(\tau))^{n+1}$.

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<u>Pontryagin classes:</u> $V$ real vect space $\rightsquigarrow V \otimes_{\mathbb{R}} \mathbb{C} = V \oplus iV$ complexification.

Similarly, $E \to B$ rank $n$ real vector bundle $\Rightarrow E \otimes \mathbb{C} \to B$ complex vect bundle.

($E \otimes \mathbb{C} \underset{\text{as real v.b.}}{=} E \oplus E$, w. mult by $i$ given by $(v_1, v_2) \mapsto (-v_2, v_1)$).
denote by $v_1 + iv_2$

<u>Lemma:</u> $\overline{E \otimes \mathbb{C}} \simeq E \otimes \mathbb{C}$

via $(v_1, v_2) \mapsto (v_1, -v_2)$
ie $v_1 + iv_2 \mapsto v_1 - iv_2$.

Hence: $c(E \otimes \mathbb{C}) = 1 + c_1(E \otimes \mathbb{C}) + \dots + c_n(E \otimes \mathbb{C}) \in H^*(B, \mathbb{Z})$

but $c_1, c_3, \dots$ are elements of order 2 since $\dots = c(\overline{E \otimes \mathbb{C}}) = 1 - c_1 + c_2 - c_3 + \dots$

<u>Def:</u> $p_j(E) = (-1)^j c_{2j}(E \otimes \mathbb{C}) \in H^{4j}(B, \mathbb{Z})$ (for $j \leq \frac{n}{2}$)

Properties:
- $p_j(E \oplus \underline{\mathbb{R}}^k) = p_j(E)$
- naturality

If $E$ is a rank $n$ <u>oriented</u> real vector bundle, then the orientation of $E \otimes \mathbb{C}$ differs from that of $E \oplus E$ by $(-1)^{n(n-1)/2}$

($(v_1, \dots v_n)$ oriented basis of $E_x$ $\rightsquigarrow$ oriented basis $(v_1, iv_1, \dots, v_n, iv_n)$ vs. $(v_1, \dots, v_n, iv_1, \dots iv_n)$.)

<u>Prop:</u> $E \to B$ oriented rank $2k$ $\Rightarrow p_k(E) = e(E)^2 \in H^{4k}(B, \mathbb{Z})$.

<u>Pf:</u> $p_k(E) = (-1)^k c_{2k}(E \otimes \mathbb{C}) = (-1)^k e(E \otimes \mathbb{C}) = (-1)^{\underbrace{k + \frac{2k(2k-1)}{2}}_{=2k^2 = \text{even}}} e(E \oplus E) = e(E \oplus E) = e(E)^2$. ∎

* This is essentially all there is, up to 2-torsion:

$\Lambda$ integral domain $\ni \frac{1}{2}$, eg $\mathbb{Q}$ (or $\mathbb{Z}[\frac{1}{2}]$)

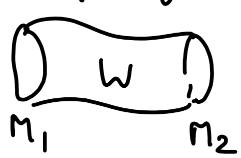
$\tilde{G}_n(\mathbb{R}^\infty)$ grassmannian of oriented n-planes $\left(\xrightarrow{2:1} G_n(\mathbb{R}^\infty)\right)$

$\Rightarrow$ Thm: $H^*(\tilde{G}_n(\mathbb{R}^\infty), \Lambda)$ = polynomial ring with generators $p_1, \ldots, p_{[n/2]}, e$ and relation $\begin{cases} e = 0 & n \text{ odd} \\ e^2 = p_{n/2} & n \text{ even} \end{cases}$

Corollary: $H^*(G_n(\mathbb{R}^\infty), \Lambda) = \Lambda[p_1, \ldots, p_{[n/2]}]$.

* Pontrjagin numbers of closed oriented 4k-manifolds are oriented cobordism invariants

ie. quantities such as $\langle \prod_j p_j(TM)^{r_j}, [M] \rangle \in \mathbb{Z}$ $\quad (\sum j r_j = k)$

$\Rightarrow$
- additive under disjoint union (clear)
- if $M = \partial W$, $W$ smooth oriented compact $(4k+1)$-mfld w/ boundary then the quantity is zero  (Pontrjagin)
- hence if  $\partial W = M_2 \amalg -M_1$ then invts equal.

Ex: oriented cobordism classes of smooth 4-mflds $\simeq \mathbb{Z}$

given by $\langle p_1(TM), [M] \rangle = 3\sigma(M) \in 3\mathbb{Z}$

$\hookrightarrow$ signature of $\cup : H^2(M,\mathbb{Z}) \times H^2(M,\mathbb{Z}) \to \mathbb{Z}$
$\alpha, \beta \mapsto \langle \alpha \cup \beta, [M] \rangle$
as a real quadratic form